PHYSICAL REVIEW E **73**, 056613 (2006)

Electromagnetic energy flux vector for a dispersive linear medium

Michael E. Crenshaw
AMSRD-AMR-WS-ST, USA RDECOM, Redstone Arsenal, Alabama 35898, USA

Neset Akozbek
Time Domain Corporation, 7057 Old Madison Pike, Huntsville, Alabama 35806, USA

(Received 3 February 2006; published 24 May 2006)

The electromagnetic energy flux vector in a dispersive linear medium is derived from energy conservation and microscopic quantum electrodynamics and is found to be of the Umov form as the product of an electromagnetic energy density and a velocity vector.

DOI: 10.1103/PhysRevE.73.056613 PACS number(s): 03.50.De, 42.50.Ct

Poynting's theorem provides the linkage between electromagnetic fields and energy by manipulating the Maxwell equations to form an energy continuity equation in terms of the divergence of an energy flux vector. A closed form for the electromagnetic energy is obtained for fields in the vacuum or in nondispersive linear media, but that is not the case for macroscopic fields in dispersive continuous media [1]. Consequently, the continuum electrodynamic treatment of the energy in dispersive media has been a long-standing issue for macroscopic quantization of the field [2]. While Poynting's theorem is based on the macroscopic Maxwell fields, Lorentz taught us that continuum electrodynamics has a microscopic basis. At the fundamental level of quantum electrodynamics, Lorentzian electrodynamics corresponds to the quantized vacuum field interacting with localized quantum oscillators. Here, we derive a macroscopic energy flux vector from energy continuity and microscopic quantum electrodynamics. The new energy flux vector appears in the Umov [3] form of an energy density multiplied by a velocity vector instead of a cross-product of the electric and magnetic fields.

The purpose of Poynting's theorem is to construct an energy continuity equation for macroscopic electric and magnetic fields from Maxwell's equations. However, continuity equations for conserved quantities can be derived under quite general conditions in a manner analogous to conservation of mass in hydrodynamics [4]. Adopting a rectangular coordinate system, x, y, and z, a differential control volume is delineated by a cube with sides of length dx, dy, and dz. The electromagnetic energy density is a scalar field $u(x,y,z,t)$. The velocity field is denoted by $\mathbf{v}(x,y,z,t)=v_x\hat{\mathbf{i}}+v_y\hat{\mathbf{j}}+v_z\hat{\mathbf{k}}$, where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are unit vectors in the direction of the respective x, y, and z axes. The property fields on the surfaces of the control volume can be related to the properties at the center of the volume by a Taylor series expansion. The energy density at each of two parallel faces of the control volume is

$$u_{x+dx/2}=u+\frac{\partial u}{\partial x}\frac{dx}{2}+\cdots, \tag{1a}$$

$$u_{x-dx/2}=u-\frac{\partial u}{\partial x}\frac{dx}{2}+\cdots. \tag{1b}$$

The energy is associated with photons and travels with the electromagnetic field. The velocity of the field at each side of the control volume,

$$(v_x)_{x+dx/2}=v_x+\frac{\partial v_x}{\partial x}\frac{dx}{2}+\cdots, \tag{2a}$$

$$(v_x)_{x-dx/2}=v_x-\frac{\partial v_x}{\partial x}\frac{dx}{2}+\cdots, \tag{2b}$$

is likewise obtained by a Taylor series expansion. The energy density and velocity at the other two pairs of faces are obtained by a similar procedure. The divergence theorem

$$\int_S u\mathbf{v}\cdot d\mathbf{A}=\int_V\left(\frac{\partial uv_x}{\partial x}+\frac{\partial uv_y}{\partial y}+\frac{\partial uv_z}{\partial z}\right)dxdydz \tag{3}$$

is then obtained from the expansions in the usual way.

The net energy that passes through the control surface in a given time must equal the change in the energy inside the control volume. This statement of conservation of energy may be written as

$$\int_S u\mathbf{v}\cdot d\mathbf{A}=-\int_V\frac{\partial u}{\partial t}dxdydz. \tag{4}$$

Combining Eqs. (4) and (3), one obtains the continuity equation

$$\nabla\cdot u\mathbf{v}=-\frac{\partial u}{\partial t} \tag{5}$$

and we can identify

$$\mathbf{S}=u\mathbf{v} \tag{6}$$

as the Umov energy flux vector. This result is constructed by energy balance, alone, and is independent of both Maxwell's equations and Poynting's theorem. In order to make the energy flux vector specific to electromagnetic radiation, we substitute

$$u=\frac{1}{8\pi}(\mathbf{E}^2+\mathbf{H}^2) \tag{7}$$

for the energy density on the left-hand side of Eq. (5), and identify

 ©2006 The American Physical Society

$$\mathbf{S} = \frac{c}{8\pi}(\mathbf{E}^2 + \mathbf{H}^2)\hat{e}_{\mathbf{k}} \tag{8}$$

as the energy flux vector in the direction of the unit vector $\hat{e}_{\mathbf{k}}$. In order to show that this result is consistent with Poynting's theorem, the electromagnetic energy density (7) is substituted into the right-hand side of Eq. (5) yielding

$$\nabla \cdot u\mathbf{v} = -\frac{1}{4\pi}\left(\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t}\right). \tag{9}$$

Substituting the Maxwell curl equations into Eq. (9) results in

$$\nabla \cdot \mathbf{S} = -\frac{c}{4\pi}[\mathbf{E} \cdot (\nabla \times \mathbf{H}) - \mathbf{H} \cdot (\nabla \times \mathbf{E})] \tag{10}$$

or

$$\nabla \cdot \mathbf{S} = \frac{c}{4\pi}\nabla \cdot (\mathbf{E} \times \mathbf{H}) \tag{11}$$

upon application of a vector identity. The Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H} \tag{12}$$

and Eq. (8) are equivalent expressions of the energy flux vector because the electric and magnetic fields are orthogonal and have the same magnitude in the vacuum.

Because the electromagnetic energy density in the vacuum, Eq. (7), is derived by Poynting's theorem, the derivation of the free-space energy flux vector (8) from the energy continuity equation does not constitute a replacement for the historic theorem. Beyond this limitation, however, the Umov energy continuity treatment of the energy flux vector is more general and is consistent with a Hamiltonian-centric view of electrodynamics, and quantum electrodynamics in particular.

In the electrodynamics of continuous media [1], the electromagnetic energy is derived by manipulating the macroscopic Maxwell equations to obtain an energy continuity equation. However, the quantum electrodynamic treatment of energy is more fundamental and the energy flux vector can be derived by coupling energy continuity to quantum electrodynamics. Starting from a microscopic quantum electrodynamic model of a linear isotropic homogeneous medium as localized quantum oscillators embedded in the vacuum, one can obtain the macroscopic effective Hamiltonian [5–7],

$$H_{enh} = \sum_{l\lambda} \hbar\, \omega_l n_l (\bar{a}_l^{\dagger}\bar{a}_l + 1/2). \tag{13}$$

Here, $\bar{a}_l$ and $\bar{a}_l^{\dagger}$ are macroscopic, or averaged, field-mode operators with the nonzero commutation relation $[\bar{a}_l, \bar{a}_{l'}^{\dagger}] = \delta_{ll'}$. The mode-dependent, and therefore frequency-dependent, effective index n_l is constructed from mode-dependent quantities that, in the continuum limit, can be associated with the vacuum, electric, and magnetic susceptibilities [7]. The Hamiltonian (13) becomes

$$H_{enh} = \frac{1}{2}\sum_{l\lambda} (n_l \hat{p}_l^2 + n_l \omega_l^2 \hat{q}_l^2) \tag{14}$$

by defining macroscopic operators

$$\hat{p}_l = -i\sqrt{\frac{\hbar\,\omega_l}{2}}(\bar{a}_l - \bar{a}_l^{\dagger}), \tag{15a}$$

$$\hat{q}_l = \sqrt{\frac{\hbar}{2\omega_l}}(\bar{a}_l + \bar{a}_l^{\dagger}) \tag{15b}$$

in analogy to their microscopic counterparts. The associated classical Hamiltonian is [5,7]

$$H_{enh} = \frac{1}{2}\sum_{l\lambda} (n_l p_l^2 + n_l \omega_l^2 q_l^2). \tag{16}$$

This is the only result that we need from quantum electrodynamics and the remainder of the discussion is purely classical. We then write the classical Hamiltonian as

$$H_{enh} = \frac{1}{2}\int_V \sum_{ll'\lambda\lambda'} \left[\sqrt{n_l n_{l'}} p_l p_{l'} \mathbf{u}_l \cdot \mathbf{u}_{l'} + c^2 q_l q_{l'} \left(\nabla \times \frac{\mathbf{u}_l}{\sqrt{n_l}}\right)\left(\nabla \times \frac{\mathbf{u}_{l'}}{\sqrt{n_{l'}}}\right)\right] dv, \tag{17}$$

where the $\mathbf{u}_l$ are the members of a complete set of orthonormal eigenfunctions of [8]

$$\nabla \times \nabla \times \frac{\mathbf{u}_l}{\sqrt{n_l}} - \frac{n_l^2 \omega_l^2}{c^2}\frac{\mathbf{u}_l}{\sqrt{n_l}} = \mathbf{0} \tag{18}$$

with periodic boundary conditions. Each mode of the field is associated with a different frequency ω_l. If the medium is dispersive, then each mode travels with a different speed c/n_l as indicated in Eq. (18). Because the enhanced energy density in a linear medium is a consequence of the reduced velocity of light, the speed, relative to c, is applied as a scale transformation [5] to the energy density from Eq. (17) and we obtain the energy flux vector

$$\mathbf{S} = u\mathbf{v} = \frac{c}{2}\sum_{ll'\lambda\lambda'} \left[p_l p_{l'} \mathbf{u}_l \cdot \mathbf{u}_{l'} + \frac{c^2}{\sqrt{n_l n_{l'}}} q_l q_{l'} \left(\nabla \times \frac{\mathbf{u}_l}{\sqrt{n_l}}\right)\left(\nabla \times \frac{\mathbf{u}_{l'}}{\sqrt{n_{l'}}}\right)\right]\hat{e}_{\mathbf{k}}. \tag{19}$$

We find the macroscopic energy flux vector in a dispersive linear medium to be

$$\mathbf{S} = \frac{c}{8\pi}(\mathbf{X}'^2 + \mathbf{Y}'^2)\hat{e}_{\mathbf{k}} \tag{20}$$

upon defining electriclike and magneticlike fields

$$\mathbf{X}' = -\sum_{l\lambda} \sqrt{4\pi} p_l(t) \mathbf{u}_l(\mathbf{r}), \tag{21a}$$

$$\mathbf{Y}' = \sum_{l\lambda} \sqrt{\frac{4\pi}{n_l}} c q_l(t) \boldsymbol{\nabla} \times \frac{\mathbf{u}_l(\mathbf{r})}{\sqrt{n_l}}. \tag{21b}$$

The new energy flux vector (20) has been derived from first principles by transforming the fundamental energy quantity of quantum electrodynamics to a macroscopic classical energy density and applying energy conservation.

The fields $\mathbf{X}'$ and $\mathbf{Y}'$ have been defined by quantities that appear as a consequence of the energy continuity relations. A more meaningful physical interpretation of the energy flux vector is obtained by relating these fields to the classical macroscopic fields. In Ref. [5], the quantum electrodynamic model of a dispersive linear medium was used to obtain just such a relation in the context of the field and material components of the energy. It was shown in Ref. [5] that the electric and magnetic fields are given by

$$\mathbf{E}' = -\sum_{l\lambda} \sqrt{\frac{4\pi}{n_l}} p_l(t) \mathbf{u}_l(\mathbf{r}), \tag{22a}$$

$$\mathbf{H}' = \sum_{l\lambda} \frac{\sqrt{4\pi} c}{n_l} q_l(t) \boldsymbol{\nabla} \times \frac{\mathbf{u}_l(\mathbf{r})}{\sqrt{n_l}}. \tag{22b}$$

The fields $\mathbf{X}'$ and $\mathbf{Y}'$ can be viewed as a combination of the respective electric or magnetic field with the appropriate reaction field. Then the energy flux vector of Eq. (20) represents the continuity of the field energy, but does not address the energy in the medium itself. This material portion of the energy is associated with the enhanced energy density and is accounted for in the reduction of the velocity of the field [5,9] rather than appearing explicitly in the energy flux vector. If dispersion may be neglected, we obtain

$$\mathbf{S} = \frac{cn}{8\pi} (\mathbf{E}'^2 + \mathbf{H}'^2) \hat{e}_{\mathbf{k}} \tag{23}$$

in the limit $\mathbf{X}' \to \sqrt{n}\mathbf{E}'$ and $\mathbf{Y}' \to \sqrt{n}\mathbf{H}'$. Further, the energy flux vector can be written as the cross-product of two vectors

$$\mathbf{S} = \frac{cn}{4\pi} \mathbf{E}' \times \mathbf{H}' \tag{24}$$

in the same dispersionless limit.

Inspecting the right-hand side of the energy continuity condition (5),

$$-\frac{\partial u}{\partial t} = -\frac{1}{2} \sum_{ll'\lambda\lambda'} \left[\sqrt{n_l n_{l'}} p_l \dot{p}_{l'} \mathbf{u}_l \cdot \mathbf{u}_{l'} + c^2 q_l \dot{q}_{l'} \left(\boldsymbol{\nabla} \times \frac{\mathbf{u}_l}{\sqrt{n_l}} \right) \cdot \left(\boldsymbol{\nabla} \times \frac{\mathbf{u}_{l'}}{\sqrt{n_{l'}}} \right) \right], \tag{25}$$

it does not appear that Eq. (25) can be expressed in closed form as the cross-product of an electriclike field and a magneticlike field in a dispersive medium. This problem is the converse trying to derive a closed form for the electromagnetic energy in a dispersive linear medium from the macroscopic Maxwell equations by Poynting's theorem, establishing the inequivalence of the Poynting and Umov vectors for dispersive linear media.

In conclusion, the microscopic quantum electrodynamic Hamiltonian is the fundamental energy quantity of electrodynamics. This Hamiltonian for a linear medium can be transformed directly into a macroscopic classical energy density in terms of electric and magnetic fields. Dispersion is naturally present in the microscopic quantum electrodynamic model of a linear medium [10] and then manifested in the macroscopic fields. Combining the Hamiltonian-based energy density and continuity of energy, we derived the electromagnetic energy flux vector in a dispersive linear medium. The new energy flux vector is found to be of the Umov form as the product of an energy density and a velocity vector rather than in the form of the Poynting vector as the cross-product of the electric and magnetic fields.

[1] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Addison-Wesley, Reading, 1960).
[2] P. W. Milonni, J. Mod. Opt. **42**, 1991 (1995).
[3] A. Chubykalo, A. Espinoza, and R. Tzonchev, Eur. Phys. J. D **31**, 113 (2004).
[4] R. W. Fox and A. T. McDonald, *Introduction to Fluid Mechanics*, 2nd ed. (Wiley, New York, 1978).
[5] M. E. Crenshaw, J. Phys. B **39**, 17 (2006).
[6] J. Knoester and S. Mukamel, Phys. Rev. A **40**, 7065 (1989).
[7] M. E. Crenshaw, Phys. Lett. A **336**, 106 (2005).
[8] D. Marcuse, *Principles of Quantum Electronics* (Academic Press, New York, 1980).
[9] M. E. Crenshaw, Phys. Lett. A **346**, 249 (2005).
[10] B. Huttner and S. M. Barnett, Phys. Rev. A **46**, 4306 (1992).